

Folding Branes

Omid Saremi^{1*}
Lev Kofman^{2,3†}
Amanda W. Peet^{1,3‡}

¹ *Department of Physics*

² *Canadian Institute for Theoretical Astrophysics
 University of Toronto
 60 St. George Street
 Toronto, Ontario
 Canada M5S 1A7*

³ *Cosmology and Gravity Program
 Canadian Institute for Advanced Research*

Abstract

We study classical dynamics of a probe Dp -brane moving in a background sourced by a stack of Dp -branes. In this context the physics is similar to that of the effective action for open-string tachyon condensation, but with a power-law runaway potential. We show that small inhomogeneous ripples of the probe brane embedding grow with time, leading to folding of the brane as it moves. We give a full nonlinear analytical treatment of inhomogeneous brane dynamics, suitable for the Dirac-Born-Infeld + Wess-Zumino theory with arbitrary runaway potential, in the case where the source branes are BPS. In the near-horizon geometry, the inhomogeneous brane motion has a dual description in terms of free streaming of massive relativistic test particles originating from the initial hypersurface of the probe brane. We discuss limitations of the effective action description around loci of self-crossing of the probe brane (caustics). We also discuss the effect of brane folding in application to the theory of cosmological fluctuations in string theory inflation.

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*omidsar@physics.utoronto.ca

†kofman@cita.utoronto.ca

‡peet@physics.utoronto.ca

1 Introduction

In this paper we investigate generic inhomogeneous solutions describing a probe D p -brane or anti-D p -brane falling in the near-horizon D-brane background geometry. A particularly interesting case occurs for D3-branes, where the background involves anti de Sitter space. The background geometry is produced by the stack of N source D p -branes. We will show that dynamics of brane motion can be reduced to that of the Dirac-Born-Infeld (DBI) type, with the action

$$S = - \int d^{p+1}x V(T) \sqrt{1 + \partial_\mu T \partial^\mu T} . \quad (1)$$

Here $T(x^\mu)$ is a scalar field (dimensionless: in units of α') and $V(T)$ is its runaway potential with no minima. The action should be understood in the truncated approximation. While first derivatives can be arbitrarily large, the action is valid only in the regime where second and higher derivatives are small. The action (1) is identical to the effective field theory action of the open string theory tachyon describing unstable non-BPS D-branes as found in e.g.¹ [1, 2, 3]. The similarity between the DBI effective action for the tachyon and that of a probe D-brane near NS5-Branes, for homogeneous solutions, was recently studied by Kutasov [4]. Some other recent works on similar topics (again for the homogeneous case) include [5, 6, 7, 8, 9].

Previous works [10, 11] on the generic inhomogeneous solution in the theory (1) were focused on the open string theory tachyon effective action, with runaway potential of exponential asymptotic form: $V(T) \simeq \exp(-T)$. A generic analytic solution was found [11] which, in particular, shows the growth of T inhomogeneities resulting in tachyon energy fragmentation and folding of T . In our paper here, we extend the method of [11] to treat generic inhomogeneities for a test D-brane or anti-D-brane freely falling in a background D-brane geometry. Folding of the field T in this context acquires a transparent interpretation, as the folding of the brane embedded in the higher dimensional spacetime background. Folding of an inhomogeneous probe D-brane should also transpire for the background geometry generated by source NS5-Branes.

The setup of moving a brane in a background spacetime generated by many branes is often an ingredient of string theory cosmological construction including models of inflation, as in e.g. [13, 14]. Usually generation of inflationary perturbations in these models are treated in the framework of four-dimensional effective theory. The effect of the brane folding which we find here is based on the full treatment of the probe brane dynamics. Thus it should be incorporated in the complete treatment of fluctuations in cosmological applications.

Before proceeding with the story of an inhomogeneous probe D-brane in the background geometry, let us recall the story of the inhomogeneous rolling tachyon. Study of the rolling homogeneous tachyon in classical string field theory has shown that, at late times, the system is described by a pressureless fluid “tachyon matter”, with no perturbative open string modes at the bottom of the potential. Turning on g_s would lead to subsequent decay of the unstable brane, mainly to massive closed string modes as in [15]. According to the proposal of [16], this process has a dual description purely in terms of the rolling open string tachyon.

It is difficult to find exact solvable CFTs describing open string tachyon dynamics for a *generic* inhomogeneous tachyon profile. Some progress in this regard was made

¹We do not attempt to be exhaustive referencing the string field theory literature here.

concerning evolution of the string theory tachyon with a plane wave tachyon profile [17]. In this case, the tachyon decays into equidistant plane-parallel singular hypersurfaces of codimension one, which were interpreted as kinks. This inhomogeneous profile is atypical, though, in the sense that fragmentation between kinks does not occur. In the general case, based on cosmological intuition, we expect both types of structures: weblike fragmentation and topological defects.

Quite apart from progress in the CFT approach, the action (1) has in fact proven to capture many striking features of tachyon condensation. The relatively simpler formulation of tachyon dynamics, in terms of the effective action (1), has therefore triggered significant interest in investigation of the field theory of the tachyon, the possible role of tachyons in cosmology, and so forth. Indeed, the end point of string theory brane inflation is annihilation of D-branes and anti-D-branes, which leads to the formation and subsequent fragmentation of a tachyon condensate [10, 11]. So the potential role of the tachyon in cosmology cannot be understood without first understanding its fragmentation.

In [10], it was found that the evolution of the tachyon field $T(t, \vec{x})$ can be viewed as a mapping $T(t_0, \vec{x}_0) \rightarrow T(t, \vec{x})$, that evolves to become multi-valued, with singularities at caustics generated. This seemed to be a generic behavior for runaway potentials. More recently, a study [11] considered the generic *inhomogeneous* tachyon field, in the region where it rolls down one side of the potential. Formation of sharp features in the tachyon energy density, due to fragmentation, was observed. The tachyon energy density pattern is reminiscent of the illumination pattern at the bottom of a swimming pool, or the web-like large scale structure of the universe. The similarity is not coincidental: the underlying mathematics has common features in all three cases. In the free streaming approximation of [10] this is just focusing of particle trajectories corresponding to higher density concentrations and, further, to the formation of caustics at the loci where trajectories cross. These features, which are related to the convergence of characteristics of the field T , are distinguished from topological defects. The full picture must incorporate both effects: formation of kinks and tachyon fragmentation in the space between them. In the context of brane inflation ending with annihilation of D-branes and anti-D-branes, the tachyon is a complex field and strings will be created. The web-like structure of fragmentation which will be superposed with the network of strings.

In this paper, we look at another situation, concerning the radial motion of a probe brane in particular string theory backgrounds. Interestingly, moving probe D-branes in the background of RR-charged or solitonic branes of various dimensions gives rise to an effective probe dynamics in which the DBI part, after some field redefinition, has the same structure as (1) – with a runaway potential of *power law* type, i.e.

$$V(T) = cT^{-\alpha} \quad (2)$$

for some $\alpha > 0$ ². For example, in the recent work [4], dynamics of a BPS test D-brane in the vicinity of stack of NS5-branes was considered, and for radial motion of the probe an action of type (1) arises, with a ‘tachyon’ potential falling off exponentially for large value of the ‘tachyon’ field. In general, the DBI part of the probe action for BPS Dp -branes probing a stack of other branes in the near-horizon region of the

²Clearly, this behavior differs from that of the usual open string tachyon potential, which is exponentially suppressed for large values of tachyon field.

background has dynamics of type (1), but with a runaway potential (2) with index $\alpha \equiv 2(7-p)/(5-p)$, which is positive, and hence sensible, for $0 \leq p < 5$.

Clearly, in these cases, there is a spacetime interpretation of the ‘condensation’ of the radial ‘tachyon’ mode, in terms of motion of the probe brane in a background geometry. Analysis of such may open a window to a better understanding of the full open string tachyon condensation itself [4].

In cosmological applications, the radial distance r to the probe brane plays the role of the inflaton field [12]. In a particular cosmological setup, cosmological scenarios based on the motion of a test anti-D3-brane in the throat region of a large number of D3-branes have recently been studied [13, 14]. Dynamics of the radial mode³ is given by an action of the above type (1) with a power-law runaway potential with $\alpha = 4$.

Inhomogeneous fluctuations of this field during inflation generate primordial cosmological fluctuations. Therefore it is important to know the time evolution for not only homogeneous, plane parallel brane motion in the anti de Sitter or more general background geometry, but generic inhomogeneous branes.

Understanding of the generic solution of the DBI type effective action may give insight to the structure of the theory.

The plan of our paper in the following. In Section 2 we introduce the supergravity description of the probe brane freely falling in the background geometry. We will show that the brane dynamics is reduced to the compact form of the DBI plus WZ effective action for a single scalar T with the power law potential. In Section 3 we make the first approximation towards the inhomogeneous solution of the non-linear equation of motion for T , the free-streaming approximation. Yet, the free-streaming approximation is not enough to calculate the stress-energy tensor of the brane. In Section 4 we construct the full analytic solution in terms of asymptotic series expansion for T . Most importantly, the next-to-leading term allows to calculate the stress-energy of the brane. The answer is transparent and allows a conjecture of a duality between a probe brane freely falling into the near-horizon background geometry (such as the AdS throat of D3-branes) and a collection of massive relativistic particles, see Section 5. In this final section we also discuss various aspects of the physics of the folding branes and its cosmological applications.

2 Probe brane in background brane geometry

The physics of a probe Dp -brane or anti- Dp -brane moving in the background of N source Dp -branes is given by the usual string-frame DBI plus Wess-Zumino action,

$$S_{\text{probe}} = S_{\text{DBI}} + S_{\text{WZ}} = -\tau_p \int d^{p+1} \sigma e^{-\Phi} \sqrt{-\det(\mathbb{P}(g_{\alpha\beta}))} \mp \tau_p \int \mathbb{P}(C_{(p+1)}) \quad (3)$$

where \mp denotes $-$ for probe anti-branes or $+$ for branes, and \mathbb{P} is for pullback.

For a (non-extremal) Dp -brane spacetime in string frame, in a standard coordinate system,

$$\begin{aligned} e^{\Phi} &= H^{(3-p)/4} \\ ds^2 &= H^{-1/2} \left(-K dt^2 + dx_{\parallel}^2 \right) + H^{1/2} \left(dr^2/K + r^2 d\Omega_{8-p}^2 \right) \\ C_{(p+1)} &= \zeta^{-1} (1 - H^{-1}) dt \wedge dx_1 \cdots \wedge dx_p \end{aligned} \quad (4)$$

³Apart from its coupling to the 4D gravity, that is.

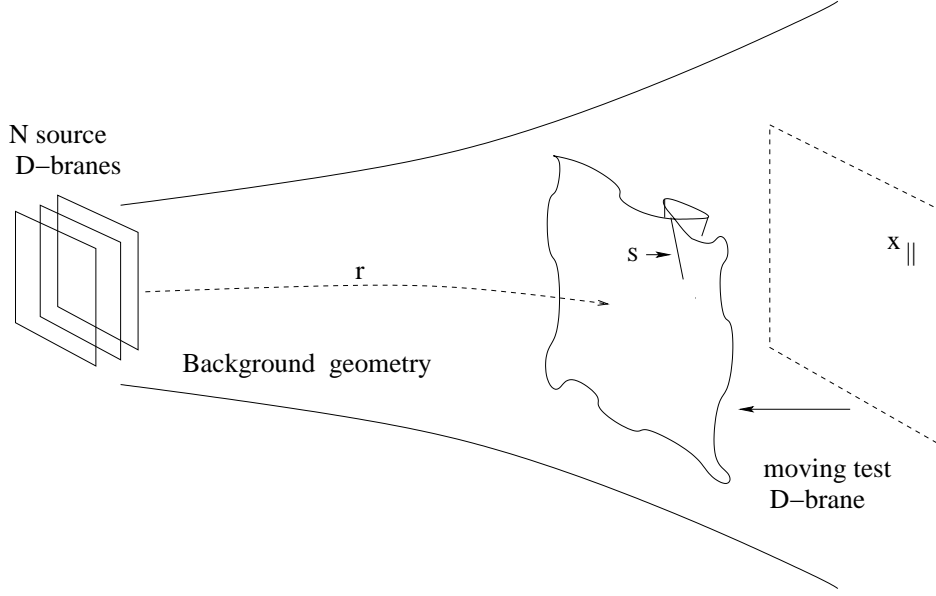


Figure 1: Sketch of probe brane moving in the background geometry.

The harmonic function is given by

$$H = 1 + \zeta \frac{g_s N \ell_s^{7-p}}{r^{7-p}} \equiv 1 + \left(\frac{R}{r} \right)^{7-p} \quad (5)$$

where $r^2 = \sum_i (x_\perp^i)^2$. The event horizon is signalled by $K(r) \rightarrow 0$, with

$$K(r) = 1 - \frac{r_H^{7-p}}{r^{7-p}} \equiv 1 - (2x)(H - 1) \quad (6)$$

where

$$x \equiv \frac{1}{2} \left(\frac{r_H}{R} \right)^{7-p} = \frac{(1 - \zeta^2)}{2\zeta} \quad \text{or} \quad \zeta = \sqrt{1 + x^2} - x. \quad (7)$$

$\zeta \in (0, 1]$ encodes the degree of non-extremality.

It follows easily that

$$S_{\text{probe}} = -\tau \int d^{p+1} \sigma \frac{1}{H} \left[\sqrt{-\det(H \bar{g}_{ij} \partial_\alpha x_\perp^i \partial_\beta x_\perp^j + \bar{g}_{\mu\nu} \partial_\alpha x_\parallel^\mu \partial_\beta x_\parallel^\nu)} \pm \zeta \frac{\partial(x_\parallel)}{\partial(\sigma)} \right] \quad (8)$$

where for convenience we use an auxiliary metric $(\bar{g}_{\mu\nu}) \equiv (-K, 1, \dots, 1, 1/K, r^2, \dots, r^2)$, which collapses to the flat metric in the BPS limit. In static gauge, where $\sigma^a = x^a$ for $a = 0, \dots, p$, this expression simplifies in the usual way to give

$$S_{\text{probe}}^{(\text{static})} = -\tau \int d^{p+1} x_\parallel \frac{1}{H} \left[\sqrt{1 + H \bar{g}_{ij} \partial_\alpha x_\perp^i \partial_\beta x_\perp^j \bar{g}^{\alpha\beta}} \pm \zeta \right] \quad (9)$$

A sketch of the probe brane in the background sourced by a stack of N Dp -branes is shown in Figure 1.

Now let us restrict to configurations in which the transverse motion is purely radial. In this case, the physics simplifies to that of just one field describing the motion of the test (probe) brane. This can be seen by performing a field redefinition to give a ‘tachyon’

$$T = \frac{R}{(p-7)} \int dH \sqrt{\frac{H}{K(H)}} (H-1)^{-(8-p)/(7-p)}. \quad (10)$$

Physically, this expression (10), in combination with the expression (5) for $H(r)$, gives the explicit connection of the radial coordinate r and the field T . In other words, T has clear geometrical interpretation in terms of the radial position of the brane embedding as $r = r(t, \vec{x}_{\parallel})$. Mathematically, in general, the relation (10) gives T in terms of hypergeometric functions of H .

The field redefinition yields

$$S_{\text{probe}}^{(\text{static}, \text{radial})} = -\tau \int d^{p+1}x_{\parallel} V(T) \left[\sqrt{1 + \bar{g}^{\alpha\beta} \partial_{\alpha} T \partial_{\beta} T} \pm \zeta \right] \quad (11)$$

where the indices run only over $\alpha, \beta = 0, \dots, p$ and the only nontrivial component of \bar{g} is the time component. In this probe action (11) for radial-only motion, the tachyon potential is given implicitly by

$$V(T) \equiv \frac{1}{H(T)}. \quad (12)$$

To obtain the explicit form of $V(T)$, it is necessary to invert the $T(H)$ expression (10) to get $H(T)$. For Dp -brane geometries where the probe is close to the stack of source branes (e.g. in the decoupling limit in the string theory context)

$$H-1 \gg 1, \quad (13)$$

the above inversion can be done relatively easily. We assume this holds in the following.

For the general case for source branes (BPS or not), with the definition

$$\alpha \equiv \frac{2(7-p)}{(5-p)} \quad (14)$$

we obtain the exact expression for $T(r)$ in terms of hypergeometric functions:

$$T(r) = \frac{(5-p)}{2R} H(r)^{1/\alpha} {}_2F_1 \left(\frac{1}{2}, \frac{1}{\alpha}; 1 + \frac{1}{\alpha}; 1 - K(r) \right) \quad (15)$$

Mathematically, this expression is monotonic in r , and is convergent for $K(r) \in [0, 1]$ (including at the endpoints) because $\alpha > 0$.

When the probe is far from the horizon, in the sense that $K(r) \sim 1^-$, the series expansion of the hypergeometric function can be employed; this is obviously also valid when there is no horizon (BPS source branes). We then get

$$T|_{\text{bps}} = \frac{(5-p)}{2R} \left(\frac{R}{r} \right)^{2/(5-p)} \quad (16)$$

Hence, large- T corresponds to small coordinate distance from the source branes. Since we are considering the large- H limit (13), this is always our regime of interest. The potential for the tachyon

$$V(T)|_{\text{bps}} = \left(\frac{2R}{(5-p)} T \right)^{-\alpha} \quad (17)$$

exhibits *power-law runaway behavior*.

In the non-BPS case, in order to obtain the near-horizon $K(r) \sim 0^+$ expansion of (15), we use a standard identity for hypergeometric functions to get

$$\begin{aligned} {}_2F_1\left(\frac{1}{2}, \frac{1}{\alpha}; 1 + \frac{1}{\alpha}; 1 - K\right) &= \frac{\Gamma(1 + 1/\alpha)\sqrt{\pi}}{\Gamma(1/2 + 1/\alpha)} {}_2F_1(1/2, 1/\alpha; 1/2; K) \\ &\quad - 2 \frac{\Gamma(1 + 1/\alpha)}{\Gamma(1/\alpha)} \sqrt{K} {}_2F_1(1/2 + 1/\alpha, 1; 3/2; K) \end{aligned} \quad (18)$$

and then, again, employ the series expansion. The upshot is that, in the near-horizon limit, the entire expression is finite all the way to the horizon. The second term in (18) is subleading at small- K , while the first term has a coefficient as a ratio of two finite Γ -functions. Amusingly, then, the non-BPS expressions for $T(H)$, and hence $V(T)$, end up being identical to the BPS case, up to a finite numerical multiplicative constant. This system therefore exhibits the same qualitative physics, as far as the runaway tachyon potential is concerned. The physically relevant qualitative difference for non-BPS branes is, of course, that the auxiliary metric on the brane worldvolume, $(\bar{g}_{\alpha\beta}) = \text{diag}(-K(T), 1, \dots, 1)$ features a non-trivial time slowdown near the horizon. In the following, we will choose to stick to the BPS case, because it can be solved exactly in an analytic asymptotic series expansion.

3 Free-streaming approximation

The equation of motion for the tachyon field follows from the action (11)

$$\square T - \frac{\partial_\mu \partial_\nu T}{1 + \partial_\alpha T \partial^\alpha T} \partial^\mu T \partial^\nu T - \frac{V_{,T}}{V} \left(1 + \zeta \sqrt{1 + \partial^\alpha T \partial_\alpha T} \right) = 0. \quad (19)$$

This equation is an example of a non-linear, partial differential equation which, as we will show, admits a relatively simple, general, inhomogeneous solution.

The energy density of the tachyon field $\rho = T_0^0$ is

$$\rho = \frac{V(T)}{\sqrt{1 + \partial_\mu T \partial^\mu T}} \dot{T}^2 + V(T) \sqrt{1 + \partial_\mu T \partial^\mu T}, \quad (20)$$

where $V = V(T)$ is given above in (17). As has been observed in [10] by solving (19) numerically, if we define an operator $P(T) = 1 + \partial_\mu T \partial^\mu T$ the field T rapidly approaches to a regime in which $P(T) \approx 0$.

Importantly, this means that in the same regime, *the Ramond-Ramond coupling is essentially irrelevant* physically. In particular, neither the \pm sign (whether the probe is a brane or an anti-brane) nor the size of ζ (the non-extremality parameter) matters! This conclusion is somewhat surprising, naively, because one would think that the

Ramond-Ramond fields would be very important near the source branes. The point is that the gauge fields become essentially irrelevant for this test-brane falling close in to the source branes, because of a strong kinetic suppression. We will nonetheless keep track of the R-R contribution to the theory on the probe.

A summary of the free-streaming tool of [10] goes as follows. In the leading approximation,

$$T(X^\mu) \approx S(X^\mu) , \quad (21)$$

where S satisfies the equation

$$\dot{S}^2 - \left(\vec{\nabla}_x S \right)^2 = 1, \quad (22)$$

where we use $t = X^0$, dot is ∂_t and the spatial derivatives are with respect to the p spatial coordinates $\vec{x} = X^\mu$ on the brane. This equation is the Hamilton-Jacobi equation for the evolution of the wave front function of free streaming massive relativistic particles. In this particle description, at some initial time t_0 we can label the position of each particle with a vector \vec{q} ; equivalently, we can say that \vec{q} parameterizes the different particles. The initial (covariant) velocity of the particle is given by $\partial^\mu S_0$. If we further define the proper time τ along each particle's trajectory, we can switch from coordinates (t, \vec{x}) to $(\tau(t, \vec{x}), \vec{q}(t, \vec{x}))$ and obtain an exact parametric solution to (22) [10]

$$\begin{aligned} \vec{x} &= \vec{q} - \vec{\nabla}_{\vec{q}} S_0 \tau \\ t &= \sqrt{1 + |\nabla_{\vec{q}} S_0|^2} \tau \\ S &= S_0 + \tau. \end{aligned} \quad (23)$$

Notice that the coordinates (τ, \vec{q}) are the proper coordinates of the probe brane. The interpretation of the solution (23) is very simple and intuitive. It tells us that the field S propagates along the trajectories of the massive relativistic particles, growing linearly with proper time. The slope of each characteristic depends only on the initial gradients of S_0 on that characteristic.

In the free streaming approximation, the denominator of the first term in the tachyon energy density expression (20) is zero, because the leading term in \dot{T}^2 is unity. Given this fact, it is clear that in order to compute the energy density near the asymptotic runaway regime, it is necessary to go beyond the free streaming approximation (22). In the next section, we compute the energy density of the tachyon by finding corrections beyond the free streaming approximation by using a working ansatz in an asymptotic series expansion.

4 The full solution

We are interested in the energy density: something the free-streaming approximation cannot give us. So let us begin here by describing the energy density qualitatively. Looking at equation (20) we see that the potential pieces are growing small, as an inverse power-law, as are the arguments of the square roots. The second term in (20) will thus rapidly become irrelevant. What we need consider is only the competition in the first term in (20) between the small potential in the numerator and the kinetic

square-root term in the denominator, all evaluated near the runaway asymptotic region of the potential. In a previous work [11], the asymptotic series solution for the exponentially decaying potentials was proposed to be $T(x^\mu) \approx S + \sum_{n=0}^{\infty} f_n e^{-(n+1)S}$. Here we generalize this result by conjecturing the following asymptotic expansion for the field T^4

$$T(x^\mu) \approx S + \sum_{n=0}^{\infty} f_n S^{-(n+1)\gamma} \quad (24)$$

In the next to the leading approximation, we have

$$T(X^\mu) \approx S + f_1 G_1, \quad (25)$$

where G_1 is of the following form:

$$G_1 = S^{-\gamma}, \quad (26)$$

where $\gamma > 0$ and

$$\alpha = \frac{\gamma + 1}{2} \quad (27)$$

Plugging the expansion (25) into equation (19) and keeping only terms linear in f_1 , gives the following equation for the function f_1

$$-S \partial^\mu S \partial^\nu S (\partial_\mu \partial_\nu f_1) + (2S \square S - 2\gamma + 2\alpha) \partial^\mu S (\partial_\mu f_1) + [(2\alpha\gamma - \gamma(\gamma + 1))S^{-1} + 2\gamma \square S] f_1 = 0, \quad (28)$$

where $\square S = \partial_\nu \partial^\nu S = -\ddot{S} + \nabla_{\vec{x}}^2 S$. This equation can be dramatically simplified by changing from (t, \vec{x}) coordinates to (τ, \vec{q}) coordinates, as defined by the characteristics of S in (23). In these coordinates, f_1 has no spatial derivatives and equation (28) reduces to

$$S f_{1,\tau\tau} + (2S \square S - \gamma + 1) f_{1,\tau} - 2\gamma \square S f_1 = 0, \quad (29)$$

where using (27) we have got rid of the term proportional to S^{-1} . Note that $\square S$ can be calculated either with respect to (t, \vec{x}) or (τ, \vec{q}) coordinates. We can further simplify this equation by introducing a new variable y

$$y = \gamma f_1 - S f_{1,\tau} \quad (30)$$

The second order differential equation (29) can be rewritten in terms of y satisfying a *first* order differential equation

$$y_{,\tau} + 2\square S y = 0, \quad (31)$$

which can be immediately integrated to give $y(\tau, \vec{q}) = y_0(\vec{q}) \exp(-2 \int^\tau d\tau' \square S)$. From this and (30)

$$f_1(\tau, \vec{q}) = f_{1i}(\vec{q}) S^\gamma \int^\tau d\tau' S(\tau', \vec{q})^{-\gamma-1} e^{-2 \int^{\tau'} d\tau'' \square S}. \quad (32)$$

(The integration constant that one gains after integrating (30) is irrelevant as it can be absorbed into S .)

⁴One can try asymptotic series with respect to other of S . However the final answer shall be independent of the choice.

To proceed further we need to estimate (32). It can be shown that $\exp(-2 \int^\tau d\tau' \square S)$ is a polynomial of order $2p$ with respect to τ such that $f_1 G_1$ is small in the asymptotic region. We start by referring back to the free-streaming parametric solution (23). Using the fact that $T \simeq S + f_1 G_1$, this shows the validity of the asymptotic series expansion (25) for runaway power-law potentials. We discuss second-order corrections in the Appendix A; these are also shown to preserve the validity of the asymptotic series expansion. Comments on arbitrary runaway potentials can be found in the Appendix B.

To compute the energy density, we need more information. Also, proper coordinates (τ, \vec{q}) are not so easily interpreted in the spacetime picture, where the brane position in static gauge is given by X^α . Therefore, let us compute the Jacobian that transforms $(\tau, \vec{q}) \rightarrow (t, \vec{x})$. To see its role, let us use the following trick. Plugging the expansion (25) into the energy density (20) we find $\rho \approx 1/\sqrt{2(\gamma f_1 + S \partial_\mu S \partial^\mu f_1)}$. The denominator here is none other than $\sqrt{2y}$. Therefore,

$$\rho \approx \frac{1}{\sqrt{2y}} = \rho_0(\vec{q}) \exp\left(\int^\tau d\tau' \square S\right). \quad (33)$$

This is precisely the expression for the energy density of free streaming relativistic particles which obey the relativistic continuity equation $\partial_\mu S (\rho \partial^\mu S) = 0$. In fact, (33) is the solution of this continuity equation in the coordinates (τ, \vec{q}) . However, there is another form of the energy density, which is equivalent to (33)

$$\rho = \rho_0(\vec{q}) / \left| \frac{\partial \vec{x}}{\partial \vec{q}} \right|, \quad (34)$$

where the denominator is the Jacobian of the $\vec{x} \rightarrow \vec{q}$ transformation. Note that we can compare expressions (33) and (34) for ρ to find a precise expression for $\square S$ and then, via (32), for f_1 . Indeed, from conservation of the energy density in a differential volume we have $d^p \vec{q} \rho_0(\vec{q}) = d^p \vec{x} \rho$, which leads to the formula (34). It is now straightforward to calculate the Jacobian from the formulas (23). In p dimensions it is a polynomial in τ of order p . This can be seen by using $\vec{x} = \vec{q} - \tau \nabla_q S_0$. We can define the symmetric tensor $D_b^a \equiv \nabla^{q_a} \nabla_{q_b} S_0$; then the Jacobian can be written in terms of the eigenvalues of D , $\{\lambda_i, i = 1 \dots p\}$, as the product $(1 - \lambda_1 \tau) \dots (1 - \lambda_p \tau)$. Using (34) for ρ , it therefore becomes clear that there is a critical proper time where the energy density blows up. In detail, at leading order in the asymptotic series the energy density is given by

$$\rho = \rho_0(\vec{q}) \prod_{i=1}^p \frac{1}{(1 - \lambda_i \tau)} \quad (35)$$

Corrections to this expression are suppressed by powers of S .

A similar behavior in the context of exponential potentials was observed in [11]. We now discuss this phenomenon in our new context of power-law runaway potentials, relevant in string theory for D-brane motion, in the next section.

5 Interpretation and Discussion

Consider the probe brane or anti-brane in the background geometry. Suppose the generic embedding is not exactly located in the hyperplane \vec{x}_\parallel , but rather given by

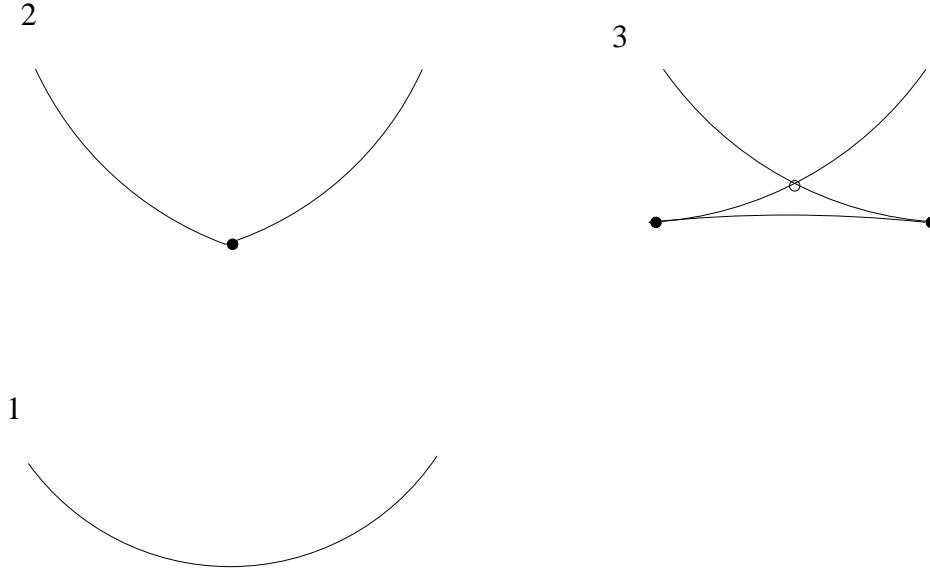


Figure 2: Folding of the brane segment from initial profile 1 through the instance 2 of the first caustic formation to typical multiple valued configuration 3.

the distance function $r = r(x_{\parallel})$. Then the following picture emerges of how the brane is moving in the background geometry. First, very quickly the brane hypersurface becomes moving in the regime of free streaming, when each point of the brane is ballistically propagating. By very quickly we mean that the test brane is in the vicinity of the near-horizon throat. Simple formulas (23), resembling the Huygens principle of geometric optics, are describing the evolution of the brane hypersurface T ; in other words, its radial position r . Geometrical rules and pictorial illustrations of the free streaming approximation are given in detail in the previous papers [10, 11].

The most important feature of brane evolution we see here is unbounded bending of its hypersurface, in other words the growth of its inhomogeneities. A simple rule stands, that the brane regions around local maxima of T are getting flatter while regions around minima are sharpening. This illustrated by the Figure 2 which shows edge-on profile of the folding brane segment around the minima of T with the most dramatic reconstruction. At some moment the brane profile acquires a discontinuity where its second derivative blows up. This corresponds to caustic formation. Formally, if we allow the free streaming approximation to work further, the brane hypersurface is self-crossing, and T goes multi valued as shown on the figure.

One may think that the multi-valuedness of the field T is related to the failure of the parametrization of the world-volume of the probe Dp -brane in static gauge. We argue that brane self-crossing is real, and accompanied by local blowup of the stress-energy tensor. Indeed, we went beyond the free-streaming approximation to describe evolution of T . While next-to-leading order terms in the series $T \simeq [S + \mathcal{O}(1/S^n)]$ have essentially negligible correction to the free-streaming propagation of T , they are fully responsible for the stress-energy tensor. However, the very simple and transparent expression for stress-energy tensor (33,34) shows that at a certain critical time and location the energy density becomes singular. The formula (33,34) is given in terms

of proper time and proper brane coordinates, and does not relate to the static gauge. We conclude that formally the effective DBI+WZ theory for generic inhomogeneous solutions for radial probe-brane motion leads to multivalued solutions, with caustics where the stress-energy tensor turns singular.

Certainly, the effective field theory description of the brane dynamics is limited by finite field gradients and stress energy components. Therefore, we have to discount the formal results beyond the points where the field gradients are large (say, larger than unity in string units, which we used here). *What happens with the probe brane around a region about to develop a caustic?* Here, we only partly address the issue and suggest some directions for future thoughts. One possibility is related to the classical physics of extended objects with sharp kink-like features. Sharp restructuring of the brane hypersurface may lead to significant radiation similar to gravitational wave flashes from cosmic string kinks. This would result in local dissipation of the stress-energy into classical radiation. Another possibility is better couched in terms of string theoretic physics. Suppose the multi-valued configuration as shown in Figure 2 begins to form. The horizontal segment of brane in configuration 3 has orientation opposite to that of the rest of the brane. That is to say, this segment now effectively describes a piece of anti-brane. Brane-antibrane segments in close proximity are unstable due to the open-string tachyon - corresponding to fundamental string stretched between the segments - which results in their annihilation.

Next, we discuss implication of our effect for cosmological applications.

The inter-brane moduli field r is often exploited as an inflaton in the string theory constructions of the brane inflation. We mention one of the realization of this idea which is close to the setup of paper, namely the inflationary model of [13]. The high dimensional construction includes a D3 brane freely falling in the background of five-dimensional anti de Sitter geometry generated by a large stack of source D3-branes. The radius r plays the role of the moduli field in the effective four-dimensional description. Under certain conditions, again, in the 4d effective description of gravity plus that moduli field inflationary regime can be realized.

One of the most important prediction of inflation is generation of scalar metric perturbations, usually associated with quantum fluctuations of the inflaton. Therefore we are interested in spatial inhomogeneities of the moduli field. In the context of our setup, these correspond to the spatial inhomogeneities of the brane embedding.

As we learned, even at the classical level brane inhomogeneities are unstable and grow. It is therefore interesting to consider combining probe-brane dynamics with inflation in $3 + 1$ dimensions. At the level when 4-dimensional gravity is introduced phenomenologically (corresponding to adding an Einstein-Hilbert term in the action by hand), in the DBI effective brane action we just have to introduce four-dimensional (quasi-) de Sitter geometry. In the equation of motion for T , this will just lead to the friction term $3H\dot{T}$ and redshifting of all spatial gradients $\partial_\mu \rightarrow \frac{1}{a}\partial_\mu$. We can conjecture that apparently folding of the brane will be suppressed at scales of the de Sitter radius $1/H$. However, it can persist at larger scales. A more quantitative description of this is left for the future.

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Appendix A

We can use a trick to derive the second-order terms in the perturbation about free streaming. Namely, we write

$$f_1 G_1 \rightarrow G_1(f_1 + f_2 G_2) \quad (36)$$

where G_2 is a power-law function $G_2 \propto S^{-\lambda}$, whose index λ to be determined.

The above trick contains nontrivial information. Indeed, for any n -th order perturbation f_n we will need to use the equation of motion for the lower-order perturbation(s), which in general yield an equation for f_n which is not identical to the equation for f_1 . Let us see this explicitly. The equations give

$$\begin{aligned} (\partial_\mu \partial_\nu f_2)(\partial^\mu S)(\partial^\nu S)S - (\partial_\mu f_2)(\partial^\mu S)[(2\alpha - 2\gamma - 2\lambda) + 2S\Box S] \\ f_2 \left[2(\gamma + \lambda)\Box S - \frac{\lambda(\lambda + \gamma)}{S^2} \right] = 0 \end{aligned} \quad (37)$$

Changing coordinates to (τ, \vec{q}) gives

$$S f_{2,\tau\tau} + [2S\Box S + (1 - \gamma - 2\lambda)] f_{2,\tau} - \left[2(\gamma + \lambda)\Box S - \frac{\lambda(\lambda + \gamma)}{S^2} \right] f_2 = 0 \quad (38)$$

Defining

$$\hat{y} \equiv -S f_{2,\tau} G_2 + (\gamma + \lambda) f_2 G_2 \quad (39)$$

we find that again the equation becomes integrable:

$$\begin{aligned} f_2 = & f_{2i}(\vec{q}) \exp \left(\int^\tau d\tau' \frac{(\gamma + \lambda)}{S(\tau')} \right) \times \\ & \left\{ \int^\tau d\tau'' \exp \left(- \int^{\tau''} d\tau' \frac{(\lambda + \gamma)}{S(\tau')} - 2 \int^{\tau''} d\tau' \Box S \right) \frac{1}{S(\tau'')^{-\lambda+1}} \right\} \end{aligned} \quad (40)$$

At long-time, $S \sim \tau$, and this equation can be integrated easily. It is then clear that $f_2 G_2$ is of the same order as f_1 . In other words, the extra piece just renormalizes f_1 . This fact holds true for *any* choice of λ .

We can obtain higher-order equations in perturbation theory by repeating our trick iteratively: substituting $f_{n-1} \rightarrow G_{n-1}(f_{n-1} + f_n G_n)$ and using lower-order equations of motion. We conjecture that at general order additional perturbations either correspond to a reparametrization of S , or a renormalization of lower-order terms, or pieces which are down by additional powers of S in an asymptotic expansion.

Appendix B

It is interesting to ask if the analysis at first order beyond the free-streaming approximation can be done for *arbitrary* runaway tachyon potential. Indeed, we show how to do this here.

We start by writing

$$T(x^\mu) \approx S + \epsilon(\tau, \vec{q}) \quad (41)$$

Next, we plug this expression into the general nonlinear equation of motion (19) for the tachyon.

In an asymptotic series expansion, ϵ should be much smaller than S , this will prove to be a self-consistent approximation. Keeping only lowest-order terms in ϵ , and using the equation of motion for S , we find a surprise: the equation for ϵ involves only derivative terms.

The next simplification occurs upon changing to proper coordinates (τ, \vec{q}) . The resulting equation of motion for ϵ becomes

$$-\epsilon_{,\tau\tau} + \epsilon_{,\tau} \left[-2\Box S + 2 \left(\frac{V_{,T}}{V} \right) \Big|_{T=S} \right] = 0 \quad (42)$$

which can be easily integrated (twice) to give

$$\epsilon(\tau, \vec{q}) = \epsilon_i(\vec{q}) \int^\tau d\tau'' \exp \left(2 \int^{\tau''} d\tau' \frac{V_{,S}}{V} \right) \exp \left(-2 \int^{\tau''} d\tau' \Box S \right) \quad (43)$$

In the long-time limit, or when initial inhomogeneities are small (which is valid here) we can do the integral involving the potential to give

$$\epsilon(\tau, \vec{q}) = \epsilon_i(\vec{q}) \int^\tau d\tau'' [V(S(\tau''))]^2 \exp \left(-2 \int^{\tau''} d\tau' \Box S \right) \quad (44)$$

As is clear from the above equation, in the long-time limit, the suppression of the next-to-leading term is controlled by the square of the tachyon potential, as was suggested in section 4. For runaway potentials, this is always small in the regime of validity of the asymptotic series expansion. In particular, note that our results here agree with the case of the power-law worked out in the body of the paper.

The energy-momentum tensor can then be written as

$$\rho(\tau, \vec{q}) \approx \frac{V(S)}{\sqrt{\Delta}} \quad (45)$$

where⁵

$$\sqrt{\Delta} \approx \sqrt{2\partial^\beta S \partial_\beta \epsilon} + (\text{higher order terms}). \quad (46)$$

We have $\partial^\beta S \partial_\beta \epsilon \sim V^2 \exp(-2 \int^\tau d\tau' \Box S)$. Substituting this into equation (45) we obtain equations (33), (34). Thus we find out that equations (33) and (34) for the energy density are valid even for arbitrary runaway potentials. This quantity $\sqrt{\Delta}$ controls the degree of kinetic suppression, of e.g. the Ramond-Ramond terms in the D-brane probe action.

⁵In proper coordinates, of course, $\partial^\beta S \partial_\beta \epsilon = -2\epsilon_{,\tau}$.

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